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Editor:

Prof. Dr. Hans-Theo Normann
Düsseldorf Institute for Competition Economics (DICE)
Phone: +49(0) 211-81-15125, e-mail: normann@dice.hhu.de

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Optimal Timing of Calling In Large-Denomination Banknotes under Natural Rate Uncertainty

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Abstract

The elimination of large-denomination banknotes is one of several options to relax the effective-lower-bound constraint on nominal interest rates. We explore timing issues associated with the calling-in of large notes from a central banker's perspective and employ an optimal stopping model to show how the volatility and the expected path of the natural rate of interest determine an optimal timing strategy. Our model shows that such a strategy can involve a wait-and-see component analogously to an optimal exercise rule for a perpetual American option. In practice, a wait-and-see component might induce a central banker not to call in large notes until the natural rate has fallen to an exceptionally low level.

JEL classification: E42, E58

Keywords: cashless economy, phase-out of paper currency, wait-and-see policy, option value

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*Heinrich Heine University Düsseldorf Department of Economics, Universitätsstraße 1, 40225 Düsseldorf, Germany, phone: +49/(0)211/81 15318, email: thomas.link@hhu.de. The author is grateful to seminar/conference participants in Düsseldorf, Rauischholzhausen, Aix-en-Provence, and Leipzig, and especially to Ulrike Neyer for the discussions and many helpful comments. The author is also especially grateful to Michalis Macrominas for a fruitful discussion on real options and games after the 33rd International Symposium on Money, Banking and Finance in Clermont-Ferrand in 2016. The impetus for this paper grew out of this discussion.

1 Introduction

It is well established that cash implies an effective lower bound (ELB) on nominal interest rates and thus a constraint for monetary policy.[1] Proposals to relax the ELB-constraint range from abolishing cash straightaway (for instance Buiter, 2009), on the one hand, to implementing measures that reduce the attractiveness of cash as an outside option in times of negative nominal interest rates on the other hand. Concrete measures of this kind include issuing banknotes with a 'magnetic strip' that can be used to enforce holders of such banknotes to pay a carry tax on currency (Goodfriend, 2000, p. 1016) or 'phasing out' large-denomination banknotes (Rogoff, 2017a, p. 57).[2] While Goodfriend's approach relies on technical ways to enforce higher costs of carry for cash artificially, Rogoff's approach aims at directly increasing the costs associated with cash hoardings like transportation, storage, and insurance costs.[3] In this paper, we take up Rogoff's proposal and consider the implied decision problem of a central banker that has the power to implement it. Our starting point is the observation that Rogoff's approach to lower the ELB is easy to implement, practical (in contrast to the other two approaches mentioned), scalable, and, most notably, that it is in fact feasible within the mandate and set of instruments of at least one major monetary authority. Evidence has recently been provided by the European Central Bank that stopped the issuance of 500-euro notes in April 2019.[4]

A central bank like the European Central Bank (ECB) with a clear mandate and the primary objective of maintaining price stability should face a relatively narrow and well-defined problem in deciding whether or not to phase out large notes – especially since it is not the complete 'abolition' of cash at stake, but just an adjustment in the

[1] See, for instance, Goodfriend (2000, p. 1007), Buiter (2009, p. 214), or Rogoff (2017a, p. 47). However, it has also been pointed out that cash is not the only reason that the ELB exists: We refer to Rogoff (2017a, p. 61) and (as also cited therein) to McAndrews (2015) for a discussion of various other frictions that had to be tackled in order for central banks to be able to effectively implement negative rates.

[2] See also Buiter and Panigirtzoglou (2003) for a detailed discussion with a comment on the feasibility and a theoretical analysis of the 'carry tax' approach. Buiter (2007a) discusses an alternative to that approach which, in principle, involves the introduction of an exchange rate between cash and central bank reserves. A detailed discussion of the approach to 'phase out' large-denomination banknotes is also presented in Rogoff (2016) and Rogoff (2017b). For a survey of several approaches to relax the ELB-constraint and a discussion of feasibility issues see, for instance, Agarwal and Kimball (2019).

[3] See Rogoff (2017a, p. 59).

[4] Agarwal and Kimball (2019, p. 44) suggest that the decision to curtail the denominational structure of banknotes has increased the scope for the ECB to lower its policy rates – simply because the physical effort of storing multiples of €500 in cash and thus the hurdle to take a flight into euro-denominated cash are higher now (which is the exact same rationale put forward by Rogoff, 2017a, p. 59).

denominational structure of banknotes. No weight should be assigned to arguments that stand against completely abolishing cash, such as the loss of privacy or of the possibility to make payments independent of information technology and access to the internet (for a central bank like the ECB, such arguments should in any case be subordinate to monetary policy goals).[5] However, even for a 'cash-averse' central banker, there is a major reason to keep issuing large banknotes: As Rogoff (2015, pp. 450–452) suggests, it is natural to assume that phasing out large notes will reduce cash demand and thus the seignorage revenue a central bank makes by issuing cash.[6] Rogoff (2015, p. 452) states that the loss of seignorage profits is in so far problematic for a central bank, as its capacity to finance itself and to thus shield its operational independence is put at stake.[7] The decision to phase out large banknotes thus involves a major dilemma for a central bank when it must trade off the benefits for monetary policy from a relaxed ELB-constraint against the loss of seignorage revenues. In this paper, a central bank's decision problem is, in principle, reduced to this dilemma because we believe that other arguments for or against the issuance of large notes, for instance, of 200- and 100-euro notes, should be less relevant for a central bank with a clear mandate like the ECB.

It is obvious that the intensity of this dilemma is state-dependent and uncertain in many dimensions with the net benefits from phasing out large banknotes dependent on the likeliness, frequency, and scale of ELB-episodes in the future as well as on the amount of forgone seignorage revenues. We focus on one key determinant and start with the hypothesis that both factors, i.e., the probability and costs of ELB-episodes in the future as well as seignorage losses, are a monotone function of the natural rate of interest. Model-based simulation results that point toward the assumption that a lower natural rate level involves a higher probability of ELB-episodes is provided by Kiley and Roberts (2017) and Chung, Gagnon, Nakata, Paustian, Schlusche, Trevino, Vilán, and Zheng (2019). These studies assess the likelihood of ELB-episodes in the United States for different states of the world and different interest rate levels and find a significant risk in some scenarios

[5]For detailed discussions of arguments in favor of and against the issuance of cash, respectively large-denomination banknotes, see, for instance, Rogoff (1998), Rogoff (2015), Rogoff (2017a) or Krüger and Seitz (2018).

[6]We use the term 'seignorage' for central bank revenue from issuing cash, noting that there are other measures of seignorage (see, for instance, Buiter, 2007b).

[7]See also Rogoff (2016, chapter 6), Buiter (2009, p. 224), Thiele, Niepelt, Krüger, Seitz, Halver, and Michler (2015, p. 10), or Krüger and Seitz (2017, chapter 4.1).

that the ELB-constraint will become binding again in the future.[8] In light of Holston, Laubach, and Williams (2017) who observe a significant fall of natural rates in the United States, Canada, the euro area, and the United Kingdom during the last three decades, it is thus natural to assume that the probability of ELB-episodes in those other regions, ceteris paribus, is also now significantly higher than it was three decades ago.[9] Altogether, with seignorage revenues typically decreasing in the interest rate level (we comment on this relationship in section 2), we build our analysis on the assumption that the net benefits from phasing out large notes and thus from relaxing the ELB-constraint are higher the lower the natural rate of interest is.

Rogoff (2016, chapter 7) discusses how a 'phase out' of large-denomination banknotes could be implemented. In principle, the implementation schemes he considers range, on the one hand, from removing the legal tender status of certain banknotes without delay or at relatively short notice (similar to the calling-in of 500- and 1,000-rupee notes in India in 2016) to, on the other hand, a 'soft' implementation version where certain banknotes are gradually removed from circulation over time by simply stopping their issuance while keeping their legal-tender status (which is the ECB's approach to phasing out the 500-euro note). In this paper, a 'tough' scheme is considered where the central bank stops the issuance of a large-denomination banknote and immediately removes its status as legal tender. We refer to this move as the 'calling-in' of the large-denomination banknote.

Our goal is to explore optimal timing issues from a central banker's perspective under three assumptions: 1) The net benefits from calling in large notes are uncertain and a function of the (stochastic) natural rate of interest. 2) The calling-in move is irreversible (for instance, because the reputational costs of reversing it are prohibitively high). 3) The move can be timed freely. With these three features, the central banker's problem of finding the optimal timing to make the calling-in move has a structure that is equivalent to the optimal exercise problem of a perpetual American option or to a firm's optimal

[8] Kiley and Roberts (2017) analyze the risk of ELB-episodes for different 'steady-state nominal interest rates'. Since they assume a 2% inflation target (see ibid., p. 337), the different steady-state nominal interest rates they consider are driven by different 'equilibrium real interest rates' (i.e., natural rates of interest). Chung, Gagnon, Nakata, Paustian, Schlusche, Trevino, Vilán, and Zheng (2019) present a related study. They show that the probability that the United States will experience an ELB-episode in the future increases with a decreasing 'neutral level of the real federal funds rate in the longer run' (see ibid., pp. 7–8). See also Yates (2004) for a review of earlier studies on the risk of ELB-episodes.

[9] See Rogoff (2017a, pp. 49–51) for a more detailed discussion on the relationship between low real interest rate levels, low monetary policy rates, and the risk of future ELB-episodes.

timing problem for an irreversible investment under uncertainty (a 'real option').[10] As pointed out by Dixit and Pindyck (1994, pp. 6–7, 153), a key feature of corresponding decisions with an option element is the 'wait-and-see' component of an optimal timing strategy. That is, under certain circumstances, optimality does not require a decision-maker to exercise a real option – analogously to a financial option – until the expected net benefits from making a move are significantly greater than zero. With regard to a central banker's decision problem this implies that even if the expected net benefits from calling in large banknotes today are greater than zero, under certain circumstances, there can be a simple reason to postpone such a calling-in move to a future date. We use a stylized optimal stopping model to show how the optimal timing of calling in large-denomination banknotes depends on the volatility and the expected path of the natural rate of interest. In the idea of concentrating on the option structure of a real policy problem, our paper is most closely related to Alvarez and Dixit (2014) who explore the euro area's 'real option' of abandoning the common currency.[11]

Section 2 formalizes a central banker's decision problem and introduces an optimal stopping model of calling in large-denomination banknotes. The model is solved in section 3.1 for the deterministic and in section 3.2 for the stochastic case. Section 4 sheds light on the determinants of optimal policy and the 'value' of the central banker's option to make a calling-in move. Section 5 presents some numerical examples to assess a central banker's wait-and-see behavior in different states of the world. Section 6 concludes.

2 An Optimal Stopping Model of Calling In Large-Denomination Banknotes

A central banker with power over the legal tender in a closed economy has the option to make a change in the denominational structure of banknotes. The central banker initially issues banknotes in two denominations, '*small*' and '*large*,' and has the authority to call in the large denomination by stopping its issuance and removing its status as legal tender

[10] See Dixit and Pindyck (1994, pp. 3–25) for a discussion on the analogy between financial and real options and their characterizing features.

[11] In solving our model and in technical regards, we primarily draw on the dynamic programming methods and solution concepts described by Dixit and Pindyck (1994).

straightaway.[12] The reputational costs of reversing such a calling-in move are prohibitively high, thus the change in the denominational structure is irreversible. The move is one-shot and can be made at any time. Time is continuous and the time horizon is $[0, \infty)$.

Cash demand and banknotes in circulation are not explicitly modeled. We just assume that *small* and *large* banknotes are only imperfect substitutes, such that cash demand depends on the denominational structure and is larger the higher the value of the largest denomination available is (for instance, because there is a specific demand for large notes as a store of value, as Fischer, Köhler, and Seitz (2004) describe). Calling in the large notes reduces cash demand and ultimately cash in circulation. For the central banker in particular, two consequences of such a calling-in move are relevant: The benefit from relaxing the ELB-constraint and a cost in the form of lost seignorage revenues.

Our way of capturing the costs and benefits from calling in large banknotes that will ultimately enter the central banker's objective function is extremely stylized: We follow the approach that Alvarez and Dixit (2014) use to formalize a currency union's decision problem of choosing the optimal timing to break up the union (they consider a potential break-up of the euro area). In principle and to put it simply, Alvarez and Dixit (2014) capture the currency union's costs and benefits from having the common currency by an exogenous process of flow utilities that is independent of the union's timing and of other future variables.[13] Although extremely stylized, this approach is convenient as it allows for a clear analysis of the effects of uncertainty over future states of the world on the decision-makers' actions and wait-and-see behavior. In this manner, we capture the central banker's net benefits from calling in large notes by the flows of utility u_t in period

[12]See also Rognlie (2016) who uses a model with two cash denominations in the context of the elimination of large denominations to lower the ELB-constraint, too. However, Rognlie (2016) considers the elimination of large denominations in the context of a New Keynesian framework and analyzes household utility under optimal monetary policy dependent on the existence of small and large cash denominations (see Rognlie, 2016, pp. 41–42). In contrast to our paper, Rognlie (2016) does not discuss issues regarding the optimal timing of eliminating large denominations.

[13]Actually, Alvarez and Dixit (2014) start with the modeling of flow benefits a member of the currency union has from belonging to the union, see Alvarez and Dixit (2014, p. 80, equation (3)).

t that are received once the calling-in move has been made. With U_t denoting the central banker's overall period-t utility we can write

$$U_t = \begin{cases} 0 & \text{for } t \in [0, T), \\ u_t & \text{for } t \in [T, \infty), \end{cases} \tag{1}$$

where $T \in [0, \infty)$ denotes the point in time when the calling-in move is made.

Since we want to describe a central banker whose main benefit from calling in large notes is a welfare gain from the relaxation of the ELB-constraint on monetary policy rates and whose main cost is the loss of seignorage revenues, we can state the central banker's net period utility from calling in large notes as

$$u_t = g - r_t - \omega, \tag{2}$$

where $g \in \mathbb{R}_{>0}$ and $\omega \in \mathbb{R}_{>0}$ are known and constant parameters and r_t denotes the natural rate of interest in period t. The natural rate is governed by an Ornstein-Uhlenbeck process (OU process) with

$$dr_t = \theta(r^{ss} - r_t)dt + \sigma dB_t, \ t \geq 0, \tag{3}$$

where B_t is Brownian motion (with $B_0 = 0$), $\theta \in \mathbb{R}_{>0}$ is the speed of reversion to a long-run mean or steady-state level $r^{ss} \in \mathbb{R}$, and $\sigma \in \mathbb{R}_{\geq 0}$ is a volatility parameter. This specific process is simple enough to allow for an analytical solution of the central banker's decision problem but it is also sophisticated enough to describe a variety of plausible empirical scenarios (we discuss different scenarios in section 3.1).[14] The two constants ω and g are level parameters that capture the costs and benefits from calling in large notes that do not depend on the state of the world, respectively on the natural rate of interest.

Letting the natural rate enter the utility function with a negative sign reflects two assumptions. The first assumption is that the potential benefits from relaxing the ELB-constraint on monetary policy rates by calling in large banknotes are larger the lower the

[14] Of course, the denominational structure itself and a lowered ELB could in turn influence the structure of the economy and in particular the natural rate, but we shall ignore such and other interdependencies and assume that the natural rate follows an exogenous process.

natural rate is. To accept this assumption it helps to consider the economics within a basic New Keynesian model as, for instance, presented in Galí (2015, chapter 3). The deviation of actual output and inflation from their natural and/or efficient levels (i.e., output and inflation gaps) in such a framework increases in the difference between the actual and the natural real rate of interest (i.e., in the real rate gap) with a positive real rate gap being associated with negative inflation and output gaps.[15] In turn, the real rate gap during ELB-episodes is larger the lower the natural rate of interest is during these periods.[16] The reason is that the central bank is unable to lower the policy rate during ELB-episodes to reduce the actual real rate of interest down to a desirable level. All in all, the welfare losses due to the inflation and output gaps during an ELB-episode and thus the benefits from relaxing the ELB-constraint are larger the lower the natural rate is.

The second assumption that is reflected in the negative sign of r_t in the central banker's period utility is that the loss in seignorage revenues in the form of central bank profits from issuing cash is larger the higher the natural rate is. A real world example can illustrate this relationship:[17] The European Central Bank's interest income from banknotes in circulation as stated in its profit and loss account is computed per convention. In principle, it is the interest the ECB earns on its share of euro banknotes in circulation, applying the rule that this share is always 8% and using a rate of return that is just the ECB's main refinancing rate (MRO rate). Consequently, the ECB's interest income from banknotes in circulation is increasing in the MRO rate and in particular it is zero at all when the MRO rate is zero (which has been the case in recent years).[18] So, returning to our model framework, the negative dependence of the central banker's utility from calling in large notes on the natural rate can be thought of as describing a world where, on the one hand, the interest income from banknotes in circulation increases in the policy rate which in turn is an increasing function of the natural rate of interest – and on the other hand, a world where

[15] See Galí (2015, p. 63, equations (22) and (23)) and also Holston, Laubach, and Williams (2017, p. 560) for a short note on this relationship.

[16] See, for instance, Galí (2015, chapter 5.4) for monetary policy under an ELB-constraint.

[17] We are grateful to Franz Seitz who pointed out this example after a seminar talk in Leipzig. See also Krüger and Seitz (2017, chapter 4.5).

[18] See European Central Bank (2019, p. A4) for these accounting rules. The ECB's share is 8% irrespective of its true value so that 92% of the euro banknote issuance are allocated to euro area national central banks that actually issue banknotes. See also European Central Bank (2019, p. A24) for the position 'interest income arising from the allocation of euro banknotes within the Eurosystem' which was zero in 2017 and 2018.

cash demand and thus currency in circulation is smaller when only small denominations are issued.[19]

Let us now consider the central banker's timing problem. We take a $t = 0$-perspective and assume that only the current level of the natural rate $r_0 = r \in \mathbb{R}$ is known such that the decision to call in large notes must be made under uncertainty over the future path of the natural rate and thus under uncertainty over the future net benefits from making a calling-in move. At this point we refer to the real options literature and in particular to Dixit and Pindyck (1994, pp. 3–25) who point out the analogy between financial options and real options, i.e., opportunities to make real investments that can be timed freely, that are irreversible, and that are made under uncertainty over future states of the world. In principle, our central banker's calling-in move can be regarded as an investment that shares the three exact same characteristic features: the move is irreversible, can be timed freely, and is made under uncertainty. So, since the central banker's timing problem has the same structure as the problem of pricing an American option or as a firm's problem of finding the optimal timing to make an investment, we use dynamic programming as described in Dixit and Pindyck (1994, pp. 93–132, 135–174) to solve this problem. In doing so, we follow Alvarez and Dixit (2014) who apply dynamic programming to value a currency union's 'real option' to break up the union.

Accordingly, we solve the central banker's problem of when to optimally call in large notes by finding the 'value' $V(r)$ of her option to make this calling-in move depending on the period-0 natural rate $r_0 = r$. We use the term 'calling-in option' from now on and measure 'value' in terms of utility such that the value of the calling-in option is the expected present value of the stream of flow utilities from calling-in large notes provided that the calling-in move will be timed optimally.[20] Determining the value function yields an optimal timing strategy for the 'exercise' of the calling-in option in the form of the rule to call in large notes as soon as the natural rate hits or falls below a certain threshold $\underline{r}$. We define the value of the calling-in option, given that the central banker follows this rule, and given that the period-0 level of the natural rate is $r \geq \underline{r}$, as supremum of the expected

[19] See, for instance, Rogoff (2016, chapter 6) for this last point in the context of a central bank's seignorage revenues.

[20] See Dixit and Pindyck (1994, pp. 99–101) for a discussion of the basic role of value functions in a dynamic programming context.

present value of the stream of flow utilities the central banker receives after having made the calling-in move at time T. With $\delta \in \mathbb{R}_{>0}$ denoting the rate at which the central banker discounts future utility, the value of the calling-in option is thus

$$V(r) = \sup_{T \geq 0} \mathbb{E}\left[\int_{T}^{\infty} (g - r_t - \omega) \cdot \exp(-\delta t) dt \;\middle|\; r_0 = r\right], \tag{4}$$

where the supremum is taken over all timings $T \geq 0$ to make the calling-in move. Note that each timing is a stopping time, i.e., a random variable that is defined by a timing strategy in the form of a rule to make the move once the natural rate hits or falls below a certain threshold. Finding the optimal threshold $\underline{r}$ and the value V of the calling-in option is an optimal stopping problem. We solve this problem in the next section.

3 Optimal Policy and Option Value

3.1 Optimal Timing and Option Value under Perfect Foresight

Before we solve the model for the case where $\sigma > 0$ in section 3.2, we consider a world with perfect foresight and thus without uncertainty over the future path of the natural rate of interest. So, for the remainder of this section, we assume $\sigma = 0$. In solving a deterministic version of the model first, we choose the same order of analysis as Dixit and Pindyck (1994, pp. 136–147) (for a generic timing problem) in order to provide some intuition on how the results of the model are driven by the non-stochastic variables, and in particular, by the anticipated path of the natural rate. For that purpose, we analyze the central banker's timing strategy in six scenarios with different paths of the natural rate.

The central banker's timing problem under perfect foresight is relatively simple and the solution is obtained as follows: Consider first the path of the natural rate. In general, with the natural rate behaving as described by equation (3), r_t conditional on $r_0 = r \in \mathbb{R}$ is Gaussian with $\mathbb{E}[r_t] = r \cdot \exp(-\theta t) + r^{ss} \cdot (1 - \exp(-\theta t))$ and $\text{Var}(r_t) = \frac{\sigma^2}{2\theta} \cdot (1 - \exp(-2\theta t))$.[21]

[21] See, for instance, Maller, Müller, and Szimayer (2009, p. 423) and Dixit and Pindyck (1994, pp. 74–78).

So, the expected natural rate is a monotone function of time. For $\sigma = 0$ the actual natural rate in period t will be equal to the expected natural rate in period t where, with $r_0 = r$,

$$r_t = r \cdot \exp(-\theta t) + r^{ss} \cdot (1 - \exp(-\theta t))\,. \tag{5}$$

Therewith, we can compute the expected (period-0) present value of the stream of flow utilities $u_t = g - r_t - \omega$ the central banker will receive after having called in the large note, given that $r_0 = r$ and given that the calling-in move will be made at time $T \geq 0$. For $\sigma = 0$, this expected value equals the actually realized value $F(r, T)$ with[22]

$$\begin{aligned} F(r,T) &= \int_T^{\infty} (g - r_t - \omega) \cdot \exp(-\delta t) dt \qquad (6) \\ &= \frac{1}{\delta} \cdot (g - r^{ss} - \omega) \cdot \exp(-\delta T) \;-\; \frac{1}{\delta + \theta} \cdot (r - r^{ss}) \cdot \exp(-(\delta + \theta)T). \end{aligned}$$

This is the central banker's objective function under perfect foresight. She maximizes (6) simply by choosing an optimal point in time $T^* \in [0, \infty) \cup \{\infty\}$ to make the calling-in move ($T^* = \infty$ means that the move will never be made). So, the optimal stopping problem (4) degenerates to

$$V(r) = \sup_{T \geq 0} F(r, T), \tag{7}$$

where $T \in [0, \infty)$ is a deterministic variable and $T^* \in [0, \infty) \cup \{\infty\}$ is thus already known in period $t = 0$ (in the general case for $\sigma > 0$, T is a random variable). Note, that as the central banker can simply choose never to exercise the calling-in option at all, its value V in terms of future utility must be bounded from below by zero.[23]

The optimal timing strategy to make the move in period T^* can also be formulated as a decision rule: do not make the calling-in move as long as the natural rate is above a certain threshold $\underline{r}$ and make the move as soon as the natural rate hits or has fallen below

[22] Here, a great technical advantage of the OU process for our purposes becomes apparent: the integral in (6) is easy to solve with r_t as defined in (5).

[23] Note, that the value of the calling-in option cannot simply be stated as maximum of F over T since such a maximum does not necessarily exist (think of F converging to zero from below for $T \to \infty$). So, we use the supremum in the formulation of the deterministic timing problem, too.

this optimal threshold. In the absence of uncertainty over the natural rate, the optimal threshold $\underline{r}$ can easily be computed by evaluating (5) at $t = T^*$ to obtain $\underline{r} = r_{T^*}$.

Therewith, the value V of the calling-in option can also be expressed as the present value of the '*exercise payoff*' $\mathcal{V}$ with

$$V(r) = \mathcal{V}(\underline{r}) \cdot \exp(-\delta T^*), \tag{8}$$

where the exercise payoff $\mathcal{V}$ at a given natural rate r is defined as the expected present value of the stream of flow utilities from calling in the large note given that the calling-in move is made at this given natural rate r with[24]

$$\mathcal{V}(r) := \mathbb{E}[F(r, T = 0)] = \mathbb{E}\left[\int_0^\infty (g - r_t - \omega) \cdot \exp(-\delta t) dt \mid r_0 = r\right] \tag{9}$$

$$= \frac{1}{\delta} \cdot (g - r^{ss} - \omega) - \frac{1}{\delta + \theta} \cdot (r - r^{ss}). \tag{10}$$

It is the specific path of the natural rate that determines whether (7) has an interior or a corner solution. In the following, we solve the model for different scenarios and show how T^* and $\underline{r}$ depend on the anticipated path of the natural rate. Recall, we have assumed that the natural rate is governed by a mean-reverting process. In the absence of stochastic movements (if $\sigma = 0$), the path of the natural rate is a deterministic and monotone function of time, and whether this function is decreasing or increasing in time depends on whether the natural rate reverts to its steady state level r^{ss} from above or from below. Therewith, the central banker's period utility from calling in large notes $u_t = g - r_t - \omega$ (as a linear function of the natural rate) features mean-reversion as well with the path of utility being inversely related to the path of the natural rate (the two constants g and ω are just level parameters). The long-run steady state level of period utility is thus just $u^{ss} = g - r^{ss} - \omega$. So, the path of period utility is monotonically increasing in time if the natural rate reverts to its steady state from above ($r_0 > r^{ss}$) and monotonically decreasing if the natural rate reverts to its steady state from below ($r_0 < r^{ss}$). In the context of the

[24]We use the term 'exercise payoff' but mean the same concept that Dixit and Pindyck (1994, p. 99) define as 'termination payoff.' For analogous timing problems see, for instance, the generic problems in Dixit and Pindyck (1994, chapter 5) or in Chevalier-Roignant and Trigeorgis (2011, chapter 9).

central banker's decision problem, these two cases describe states of the world where the ELB-constraint becomes more, respectively less, relevant as time evolves.

Now, for the following analysis, we define two regions, $\mathcal{A}$ and $\mathcal{B}$, of the parameter space where $\mathcal{A} : u^{ss} > 0$ and $\mathcal{B} : u^{ss} \leq 0$. A positive steady state of utility means that in the long run the benefits from calling in large-denomination banknotes will be greater than the costs, a negative steady state means that the costs will exceed the benefits in the long run. In each of the two regions, we consider three scenarios I, II, and III with an initially *'high,'* an initially *'near-steady-state,'* and an initially *'low'* natural rate reverting to its steady state, respectively.

Scenario $\mathcal{A}.I$ $(r_0 = r > g - \omega > r^{ss})$ This scenario describes a world where in the long run, the benefits from calling in large notes are greater than the costs. With our interpretation of the costs and benefits this means that, in the long run, the central banker's benefits from relaxing the ELB-constraint exceed the costs in the form of lost seignorage revenues. But with a relatively high natural rate in period $t = 0$, this scenario also describes a world where ELB-issues are at first irrelevant and only gain importance as time evolves and the natural rate decreases to a relatively low steady state which involves a relatively high 'risk' of policy rates hitting the ELB. Formally, this is reflected in the period-0 flow utility $u_0 = g - r_0 - \omega$ which is negative for $r_0 > g - \omega$ but increases as time evolves. What we want to show is that the central banker will wait to make the calling-in move until the natural rate has fallen to a sufficiently low level – although moving earlier would already yield a positive exercise payoff.

Now, if the steady state of utility u^{ss} is strictly positive, we can derive the optimal timing T^* of the calling-in move as well as the level $\underline{r}$ of the natural rate at which the option is optimally exercised simply by maximizing F over $T \in [0, \infty)$. In this case,

$$V(r) = \sup_{T \geq 0} F(r, T) = \max_{T \geq 0} F(r, T). \tag{11}$$

The first-order condition for an interior maximum is implied by

$$\frac{\partial F(r, T)}{\partial T} = -(g - r^{ss} - \omega) \cdot \exp(-\delta T) + (r - r^{ss}) \cdot \exp(-(\delta + \theta)T), \tag{12}$$

and reads

$$(g - r^{ss} - \omega) \cdot \exp(-\delta T) = (r - r^{ss}) \cdot \exp(-(\delta + \theta)T). \tag{13}$$

This yields the unique interior solution[25]

$$T^* = \frac{1}{\theta} \cdot \ln\left(\frac{r - r^{ss}}{g - r^{ss} - \omega}\right) \tag{14}$$

which implies

$$\underline{r} = r_{t=T^*} = g - \omega. \tag{15}$$

This $g - \omega$ threshold is critical: If the natural rate is below the threshold $g - \omega$ it is so low that the central banker's period utility u_t is positive (with $\underline{r} = g - \omega$ it is clear that $u = g - r' - \omega > 0$ for all $r' < \underline{r}$). With our interpretation of the central banker's costs and benefits we can reformulate this statement: A natural rate below the threshold $g - \omega$ is so low that ELB-issues outweigh seignorage losses. This justifies the decision rule that is implied by $\underline{r}$, which is to make the calling-in move as soon as the period utility u_t becomes non-negative (recall that without stochastic fluctuations, the period utility u_t will steadily approach its steady state u^{ss} which in this scenario is positive, so once it has become non-negative, the period utility will stay positive forever, given that $\sigma = 0$).

If the central banker times the calling-in move optimally, respectively follows the decision rule to make the move once the natural rate hits $\underline{r}$, she will receive only positive flows of utility. Consequently, the present value of these flows, i.e., the value V of the calling-in option, will be strictly positive.[26] So, the central banker could still receive a positive exercise payoff if she moved 'somewhat' earlier before the natural rate hits $\underline{r}$.

All in all, this scenario describes a situation where deferring the calling-in move is rational although the payoff from making the move 'somewhat' earlier would already be

[25] With $\frac{\partial^2 F(r,T)}{\partial T^2} = \delta(g - r^{ss} - \omega) \cdot \exp(-\delta T) - (\delta + \theta) \cdot (r - r^{ss}) \cdot \exp(-(\delta + \theta)T)$ it easily checked that $\partial^2 F(r, T^*)/\partial T^2 < 0$ for $r > g - \omega > r^{ss}$ and that the global maximum of F on $[0, \infty)$ is in fact in $T^* > 0$.

[26] This can easily be checked by considering $V(r) = F(r, T^*) = \mathcal{V}(\underline{r}) \cdot \exp(-\delta T^*) = \left(\frac{1}{\delta}(g - r^{ss} - \omega) - \frac{1}{\delta+\theta}(g - r^{ss} - \omega)\right) \cdot \exp(-\delta T^*) = \frac{\theta}{\delta^2 + \delta\theta} \cdot u^{ss} \cdot \exp(-\delta T^*)$ which is strictly greater than zero if $u^{ss} > 0$.

greater than zero. The reason for this deferral is the anticipated decline of an initially high natural rate to a relatively low steady state in the future – which describes a world where ELB-issues are initially irrelevant but are only gaining importance over time. As making the calling-in move at a relatively high natural rate level initially would lead to negative period utilities, waiting until the period utility becomes greater than zero increases the central banker's overall payoff.

The central banker's tendency to defer the exercise of the calling-in option, that is, to wait due to the anticipated reversion of the natural rate to its steady state is reflected in the length of the interval between the optimal threshold $\underline{r}$ and the '*break-even threshold*' $\hat{r}$ defined as the natural rate below which the central banker would make the calling-in move in a situation where she would have to decide between making the move now or never. The break-even threshold is implicitly defined by $F(r_0 = \hat{r}, T = 0) = 0$ which yields

$$\hat{r} = g - \omega + \frac{\theta}{\delta} \cdot (g - r^{ss} - \omega) = g - \omega + \frac{\theta}{\delta} u^{ss}. \tag{16}$$

The representation of the break-even threshold $\hat{r}$ in (16) illustrates that the larger the benefits from calling in large notes are in the long run, i.e., the larger u^{ss} is, the earlier the central banker could make the move without incurring a negative exercise payoff.

Thus, if $r_0 = r \in (\underline{r}, \hat{r})$, the period-0-value of the payoff from exercising the calling-in option at $T = 0$ is already strictly greater than zero, but optimality requires the central bank to defer the calling-in move to T^* until also the period flow utility u_t exceeds zero (recall that $u_{t=T^*} = g - r_{t=T^*} - \omega = 0$, with $r_{t=T^*} = \underline{r} = g - \omega$). So, the move is deferred to avoid negative streams of period utilities.

Also the next two scenarios describe a world where the long-run benefits from calling in large notes are greater than the costs ($u^{ss} > 0$). But with $r_0 < g - \omega$ in both scenarios, the natural rate is and will remain so low that the short-run benefits from calling in large notes are also so large that optimality requires the central banker to make the calling-in move without delay in period $t = 0$. So, the next two scenarios describe an economic environment where ELB-issues will be relevant from the outset and forever. Trivially, the decision rule to make the move as soon as the natural rate hits or has fallen below the threshold $g - \omega$ also applies in the next two scenarios.

Scenario $\mathcal{A}.II$ $(g-\omega \geq r_0 = r > r^{ss})$ The optimal timing of the calling-in move in this scenario is $T^* = 0$. This is a corner solution of $\max_{T\geq 0} F(r,T)$ with $F(r,T=0) > 0$ and $\partial F(r,T)/\partial T < 0 \; \forall \; T \in (0,\infty)$. The reason that the move should be made in period $t=0$ is that period utility u_t is positive from the outset (since $r_0 < g-\omega$).

Scenario $\mathcal{A}.III$ $(g-\omega > r^{ss} \geq r_0 = r)$ As in scenario $\mathcal{A}.II$, the optimal timing of the calling-in move is $T^* = 0$. Again, this is a corner solution of $\max_{T\geq 0} F(r,T)$ with $F(r,T=0) > 0$ and $\partial F(r,T)/\partial T < 0 \; \forall \; T \in [0,\infty)$. The difference to the previous two scenarios is that the natural rate reverts to its steady state from below. One could interpret this scenario as describing a world during or after a financial or an economic crisis where a relatively low natural rate produces a severe and pronounced ELB-episode. As time evolves, this severe episode will find an end, but the relatively low steady state of the natural rate implies that ELB-issues will remain relevant forever.

Scenario $\mathcal{B}.I$ $(r_0 = r > r^{ss} > g-\omega)$ With a high natural rate r_0 and a relatively high steady-state level r^{ss}, this scenario describes a world in which the ELB-constraint is and will be of no importance such that relaxing it would be useless. Neither the short-run benefits nor the long-run benefits from making the calling-in move will exceed the costs. With $r_0 = r > r^{ss} > g-\omega$, the period flow utility u_t from calling-in large notes will never be positive so that $F(r,T) < 0 \; \forall \; T \geq 0$ and thus $V(r) = 0$. The calling-in option will never be exercised.

Scenario $\mathcal{B}.II$ $(r^{ss} \geq r_0 = r > g-\omega)$ As in scenario $\mathcal{B}.I$, the natural rate will always remain so high that the period utility u_t from calling in large notes will always be negative. Hence, $F(r,T) < 0 \; \forall \; T \geq 0$ and $V(r) = 0$. The calling-in move will never be made.

Scenario $\mathcal{B}.III$ $(r^{ss} > g-\omega \geq r_0 = r > -\infty)$ This scenario describes a world where the benefits from making the calling-in move exceed the costs in the short run ($r_0 < g-\omega$ such that $u_0 > 0$) but not in the long run ($u^{ss} < 0$). For instance, large-scale financial or economic crises could feature such exceptionally low natural rates. With monetary policy rates that have reached the ELB in such a scenario, calling in large notes could be a rational move even if this decision entailed long-run losses (as long as future losses are discounted – which we assume by setting $\delta > 0$). The condition for making the move is that the short-run benefits are sufficiently large, which is only the case for exceptionally low natural rates such that the short-run period utility from calling in large notes is significantly greater than zero. The condition for a significantly positive utility u_t is that the natural rate is significantly below the threshold $g-\omega$. We can state this more precisely by considering the break-even level $\hat{r}$ again as defined in (16) as $\hat{r} = g-\omega+\frac{\theta}{\delta}u^{ss}$. Recalling that $u^{ss} < 0$ in this scenario, it is clear that the break-even threshold $\hat{r}$ is lower, the smaller the steady state u^{ss} is. Since the natural rate increases as time evolves, optimality requires the central banker to make the move without delay in period $t = 0$ if $r_0 \leq \hat{r}$. Thus, the rule of whether or when to make the calling-in move in a world where this move implies long-run losses ($u^{ss} < 0$) is implied by the optimal threshold $\underline{r} = \hat{r}$: move immediately if $r_0 \leq \hat{r}$, and do not move if $r_0 > \hat{r}$.[27]

3.2 Optimal Exercise Rule and Option Value under Uncertainty

We now solve the central banker's optimal stopping problem (4) in a world without perfect foresight where the path of the natural rate of interest is uncertain, i.e., where $\sigma > 0$. The solution approach we use is taken from Dixit and Pindyck (1994) which is our main reference in technical regards (as far as possible, we use the [shorthand] notation proposed therein).[28] In addition, we refer to Øksendal (2013) for some basic methods of Itô calculus that are used here and Lebedev (1965) for the differential equation and the solution we obtain.

So, following Dixit and Pindyck (1994) in this technical regard, we use dynamic programming based on Bellman's principle of optimality to solve the optimal stopping problem

[27] We assume implicitly that the calling-in option is also exercised if $r_0 = \underline{r} = \hat{r}$. Moreover, we have not considered the case $r^{ss} = g - \omega$ so far which implies $u^{ss} = 0$. With (16) it becomes clear that in this case $\hat{r} = g - \omega = r^{ss}$ so that the option is exercised never if $r_0 = r > r^{ss}$ and exercised at $t = 0$ if $r_0 = r < r^{ss}$. For the case $r_0 = r = r^{ss} = g - \omega$ implying $F(r, T) = 0 \ \forall \ T \in [0, \infty)$ we break ties in favor of the calling-in option being exercised at $t = 0$.

[28] For instance, Dixit and Pindyck (1994) describe how to solve the optimal stopping problem of a firm that has the option to make a real investment under uncertainty over the future value of that investment. Inter alia, they show how to use dynamic programming to solve the firm's problem when the evolution of the project value is described by geometric Brownian motion (ibid., pp. 140-147) or by a mean-reverting process that has an absorbing state (ibid., pp. 161-167) (note, the process defined by (3) that we use to describe the path of the natural rate has no absorbing state).

(4).[29] Accordingly, the value function $V(r)$ and the optimal threshold $\underline{r}$ can be obtained by solving the Bellman equation

$$\delta V dt = \mathbb{E}\left[dV\right] \tag{17}$$

for V where (17) holds for all levels of the natural rate r that are so high that optimality requires the central banker to keep on issuing large banknotes – i.e., for all $r \in [\underline{r}, \infty)$.[30]

Finding V and $\underline{r}$ is a free boundary problem.[31] We solve this problem by using the solution approach described by Dixit and Pindyck (1994, pp. 95–114, 130–132) and in particular by ibid. (pp. 140–147). Accordingly, we start by using the Itô formula to write the Bellman equation (17) as the homogeneous ordinary differential equation

$$\frac{1}{2}\sigma^2 V'' + \theta(r^{ss} - r)V' - \delta V = 0 \tag{18}$$

(see appendix A for the detailed derivation).[32]

By introducing economically meaningful boundary conditions we can solve equation (18) for V and obtain $\underline{r}$.[33] We assume that the solution of (18) must satisfy three boundary conditions – two left and one right boundary condition. The two left boundary conditions we apply are standard in the literature where they are often referred to as 'value-matching condition' and 'smooth-pasting condition': Referring to Dixit and Pindyck (1994, pp. 109, 130–132, 141) and Peskir and Shiryaev (2006, chapters 8, 9) for a further discussion of

[29] See Dixit and Pindyck (1994, p. 100) for a discussion of Bellman's principle of optimality in the context of the valuation of investment projects.

[30] For a discussion of related Bellman equations that solve optimal stopping problems in continuous time with an infinite time horizon see Dixit and Pindyck (1994, pp. 101-114) and in particular ibid. (p. 140, equation (7)) where a Bellman equation is discussed that is equivalent to the Bellman equation we have. For a discussion of why a value function in an infinite-time-horizon setting does not explicitly depend on time see, for instance, Dixit and Pindyck (1994, p. 107). For a Bellman equation in the context of a monetary union's problem of when to optimally break up the union see Alvarez and Dixit (2014, p. 81, equation (9)).

[31] See Dixit and Pindyck (1994, p. 109).

[32] Although the context of these papers is far away from our subject, we refer to Parlour and Walden (2009, p. 13, equation (14)), Garlappi and Yan (2011, p. 819, equation (A2)), and Suzuki (2016, p. 39, equation (42)) who obtain equivalent/similar differential equations in their respective valuation problems with state variables that also follow an Ornstein-Uhlenbeck process, respectively.

[33] See Dixit and Pindyck (1994, p. 109) for a short note on the rationales of respective boundary conditions in optimal stopping problems in an economic context.

the concepts, respectively the rationales, of these conditions, we use the value-matching condition

$$V(\underline{r}) \overset{!}{=} \mathcal{V}(\underline{r}) \tag{19}$$

that requires the value of the calling-in option V to equal the option's exercise payoff $\mathcal{V}$ in the moment the option is exercised, i.e., when $r = \underline{r}$, and the smooth-pasting condition

$$V'(\underline{r}) \overset{!}{=} \frac{\partial \mathcal{V}(\underline{r})}{\partial \underline{r}} \tag{20}$$

that requires the value function V at $\underline{r}$ to have the same slope as the exercise payoff function $\mathcal{V}$ at $\underline{r}$.[34]

In addition to these two left boundary conditions, we introduce a right boundary condition arguing that this is a natural assumption with respect to the calling-in option valued at a very high natural rate of interest: We require that

$$\lim_{r \to \infty} V = 0 \tag{21}$$

and thus capture the intuition that the expected present value of the exercise payoff (and therewith the value of the calling-in option) should be smaller the longer it will presumably take until the natural rate hits the optimal exercise threshold $\underline{r}$.[35]

The value function must also satisfy the non-negativity constraint

$$V \geq 0 \tag{22}$$

which, trivially, just reflects that the central banker has the option to issue large notes forever. Now, it is straightforward to use this condition together with the boundary conditions to obtain a particular solution of (18). We have placed the single steps in the appendix and summarize the results in the next proposition:

[34] For an application of the value-matching and smooth-pasting condition in the context of a currency union's optimal stopping problem of when to break up the union see Alvarez and Dixit (2014, p. 81).

[35] To see this, consider equation (8) stating that $V(r) = \mathcal{V}(\underline{r}) \cdot \exp(-\delta T^*)$ for the case of $\sigma = 0$ and recall that T^* increases in r. See also Garlappi and Yan (2011, p. 819) who have an equivalent right boundary condition.

Proposition 1. *A particular solution of the Bellman equation* (18) *that solves the central banker's optimal stopping problem* (4) *subject to* (19), (20), (21), *and* (22) *is given by*

$$V(r) = c_1 \cdot H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma} \cdot (r - r^{ss})\right), \tag{23}$$

with

$$c_1 \;=\; \frac{1}{\delta+\theta} \cdot \frac{\sqrt{\theta}\cdot\sigma}{2\delta} \cdot \frac{1}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma}\cdot(\underline{r} - r^{ss})\right)}, \tag{24}$$

and with $\underline{r}$ *being implicitly defined by*

$$\frac{1}{\delta+\theta} \cdot \frac{\sqrt{\theta}\cdot\sigma}{2\delta} \cdot \frac{H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma}\cdot(\underline{r} - r^{ss})\right)}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma}\cdot(\underline{r} - r^{ss})\right)} = \frac{1}{\delta}(g - r^{ss} - \omega) - \frac{1}{\delta+\theta}(\underline{r} - r^{ss}), \tag{25}$$

where $H_\nu(z)$ *denotes a Hermite function (as defined, for instance, in Lebedev, 1965, p. 285). Thereby, the value function is defined piecewise: The value function* $V(r)$ *is given by* (23) *for all* $r \in [\underline{r}, \infty)$. *For all* $r < \underline{r}$, *immediate exercise of the calling-in option is optimal such that the value function for all* $r < \underline{r}$ *is just defined as* $V(r) = \mathcal{V}(r)$.

Proof. See appendix (note that since our main intention is to show that the central banker's problem of finding the optimal timing of calling in large banknotes has a structure that is equivalent to the structure of an option valuation problem, we just prove the existence and not the uniqueness of our solution). □

In the next section, we discuss the determinants of the optimal exercise threshold and the value of the calling-in option in detail. In section 5, in order to obtain illustrative results, we solve equation (25) numerically for specific parameter values and use the resulting approximations of $\underline{r}$ to compute and analyze the value V of the calling-in option in different scenarios.

4 Determinants of Optimal Policy and Option Value

4.1 Option Value and Measures of the Central Banker's Tendency to Wait and See

In the following, we first make some preliminary remarks on the value of the calling-in option and on different measures of the central banker's tendency to wait and see. Subsequently, in sections 4.2 to 4.5, we discuss the determinants of the break-even threshold $\hat{r}$, the optimal exercise threshold $\underline{r}$, and the option value.

The value of the calling-in option consists of two components. Keeping the terminology commonly used for financial options, we refer to these two components as 'intrinsic' and 'time value'. The intrinsic value (IV) is the positive part of the payoff from immediately making the calling-in move, that is, $IV(r) = \max\{\mathcal{V}(r), 0\}$. The time value ($TV$), is defined as $TV(r) = V(r) - IV(r)$. The time value is a measure of the central banker's tendency to wait and see instead of calling in large banknotes at the first opportunity where a non-negative exercise payoff could be realized.[36]

Two factors can add time value to the calling-in option: The first one is related to the expected path of the natural rate. If the natural rate is expected to decrease as time evolves, the central banker will expect the benefits from calling in large notes to increase over time. If, additionally, the long-run benefits from calling in large notes are positive, i.e., if $u^{ss} > 0$, there can be a reason to delay the calling-in move until the short-run benefits from removing large notes are sufficiently large – even if the immediate exercise of the calling-in option yields a positive payoff (as illustrated in scenario $\mathcal{A}.I$ in section 3.1). The second factor that adds time value to the calling-in option, of course, is uncertainty over the future path of the natural rate. This uncertainty is reflected in $\mathrm{Var}(r_t)$ for $t > 0$ and thus depends on the volatility parameter σ and on the speed of mean-reversion θ (recall, as noted in section 3.1, that $\mathrm{Var}(r_t) = \frac{\sigma^2}{2\theta} \cdot (1 - \exp(-2\theta t))$).

The time value reflects the central banker's tendency to wait and see and thus crucially determines the optimal timing of the calling-in move. Below, we evaluate this tendency to wait and see in different scenarios in terms of the option's time value and with the following

[36]See, for instance, Hull (2019, p. 284).

other measures:[37] The tendency to wait and see due to an expected decline of the natural rate and thus due to increasing expected benefits from calling in large notes is reflected in the difference between the break-even threshold $\hat{r}$ and the optimal threshold under perfect foresight $\underline{r}|_{\sigma=0}$. The tendency to wait due to uncertainty over the future path of the natural rate is reflected in the difference between the optimal threshold under perfect foresight and the actually optimal threshold $\underline{r}$. And obviously, the 'overall' tendency to wait and see is reflected in the length of the interval $[\underline{r}, \hat{r}]$. So, in the following, we analyze the variables (in that order) $\hat{r}$, $\underline{r}|_{\sigma=0}$, $\underline{r}$, and TV with respect to their dependency on σ, θ, and u^{ss}. Note, that we use the variance of r_t (conditional on $r_0 = r$) as a measure of uncertainty over the natural rate in period t and use the terms 'long-term uncertainty' in this respect for 'large' t and 'short-term uncertainty' for 'small' t. With this classification, uncertainty (over both short and long horizons) is increasing in the volatility parameter σ and decreasing the speed of mean-reversion θ.[38]

4.2 Determinants of the Break-Even Threshold

Uncertainty The way in which we specify our model implies that the break-even threshold $\hat{r} = g - \omega + \frac{\theta}{\delta} \cdot (g - r^{ss} - \omega) = g - \omega + \frac{\theta}{\delta} u^{ss}$ (as stated in (16)) does not depend on the volatility parameter σ and is thus independent of uncertainty over the future path of the natural rate. The advantage of this specification is that the effect of σ on the optimal exercise threshold $\underline{r}$ is – as intuition suggests – monotonic (we discuss this effect in section 4.4).[39] To see that $\hat{r}$ is independent of σ, recall that $\hat{r}$ is determined by the expected path of period utility which, in turn, is a linear function of the expected path of the natural rate. There is no link between $\hat{r}$ and σ because the two expected paths are independent of the volatility parameter σ.

[37] See Alvarez and Dixit (2014, pp. 85–86) for a discussion of different measures of option value.

[38] While it is obvious that $\frac{\partial \text{Var}(r_t)}{\partial \sigma} > 0$, it is not immediately clear why the second statement that $\frac{\partial \text{Var}(r_t)}{\partial \theta} < 0$ holds. To see why it holds, consider $\frac{\partial \text{Var}(r_t)}{\partial \theta} = \frac{\sigma^2}{2\theta^2} \cdot (\exp(-2t\theta) \cdot (2t\theta + 1) - 1)$. To show that this expression is less than zero, it is sufficient to show that $\exp(-2t\theta) \cdot (2t\theta + 1) < 1$. Taking the $ln(\cdot)$, rearranging and using the substitution $x = 2t\theta$ the last inequality can be transformed to $ln(x + 1) < x$ which, in turn, can easily be proved.

[39] For instance, a non-monotonic effect of volatility on a respective optimal threshold is described in Alvarez and Dixit (2014).

Steady-State Utility In a world where the central banker would lack the flexibility to time the calling-in move freely being in a situation where she would have to choose between making the move immediately or issuing large notes forever, a relatively high steady state of utility (i.e., the prospect of large eternal long-run benefits from calling in large notes) could make her willing to call in large notes even if she had to accept losses in the short run. Thus, under special circumstances, a central banker who would have to make the decision whether to call in large notes now or never while expecting large long-run benefits from calling in large notes would be willing to make the move already and even at a relatively high natural rate which would involve short-run losses. Thus, the break-even threshold $\hat{r}$ (being the relevant threshold for a central banker in a hypothetical now-or-never situation) is higher the higher the steady-state utility is. Formally, this relationship is captured by (16) which illustrates that a positive steady-state utility induces a markup over the level $g - \omega$ which is the threshold below which natural rates result in a positive period utility.[40]

Speed of Mean-Reversion In the following, we consider a world where the central banker lacks the flexibility to time the calling-in move freely. Concretely, we assume that the central banker can choose between making the calling-in move immediately or issuing large notes forever. We consider two scenarios in this 'now-or-never' world to illustrate that a higher speed of mean-reversion either increases or decreases $\hat{r}$, dependent on whether utility has a positive or a negative steady state (which, formally, is immediately clear with $\hat{r} = (1 + \frac{\theta}{\delta}) \cdot u^{ss} + r^{ss}$). Since $\hat{r}$ does not depend on σ (see above), we let, without loss of generality, $\sigma = 0$ in the following two scenarios and assume perfect foresight with respect to the future path of the natural rate of interest. The scenarios differ in the steady state of utility with the first scenario characterized by $u^{ss} < 0$ and the second by $u^{ss} > 0$. The present in both scenarios is $t = 0$ with the natural rate at this point in time just having hit the break-even level, i.e., $r_0 = r = \hat{r}$.

So, consider the first scenario where period utility has a negative steady state while $r_0 = r = \hat{r}$ which implies that the natural rate reverts to its steady state from below.

[40] The break-even threshold can also be written as a weighted sum of steady-state utility and the steady-state natural rate which acts as a level parameter such that $\hat{r} = (1 + \frac{\theta}{\delta}) \cdot u^{ss} + r^{ss}$. This makes clear that the exercise of the calling-in option in a now-or-never situation in a $u^{ss} < 0$-scenario can only be optimal at natural rates that are strictly below their steady-state level.

With respect to the anticipated path of period utility, the future from a $t = 0$ perspective can be partitioned into two consecutive phases: the first phase is one of relatively low natural rates (since $\hat{r} < r^{ss}$) and thus a phase where the calling-in of large banknotes is beneficial to the central banker (period utility is positive during this phase because the ELB-constraint is relevant). The second phase features relatively high natural rates such that (due to the irrelevance of the ELB-constraint) having stopped issuing large notes now involves losses for the central banker (due to forgone seignorage revenues), i.e., period utility is negative during the second phase. It is clear that making the calling-in move is only optimal if the short-run benefits from calling in large notes outweigh the long-run losses. That is, the move is only optimal if the first phase with positive period utilities is sufficiently long and/or if the period utilities during this phase are sufficiently high. However, the first phase is shorter the higher the speed of mean-reversion θ is. So, the higher θ, the higher are the short-run benefits the central banker requires to make the calling-in move and thus the lower the break-even threshold $\hat{r}$ (see the red lines in figure 1(a) for an illustration).

Consider now the second scenario with period utility having a positive steady state. Again, the time after the calling-in move is made can also be partitioned into two consecutive phases, a phase of negative period utilities followed by a phase of positive period utilities (recall that $\hat{r} > g - \omega > r^{ss}$ implying that the natural rate reverts to its steady state from above in that case). Now, with the prospect of eternal long-run benefits from calling in large notes (that will be received once the natural rate has fallen below the level $g - \omega$, i.e., when ELB-issues outweigh seignorage losses), the central banker will accept making short-run losses, that is, she will accept negative period utilities during the first phase of relatively high natural rates. The cumulative short-run losses are smaller the shorter the first phase is, that is, the higher the speed of mean-reversion is. Conversely, the shorter the first phase is, the higher the period losses the central banker accepts during the first phase. Thus, the higher θ, the higher the break-even threshold $\hat{r}$ (see the red lines in figure 1(b) for an illustration).

4.3 Determinants of the Optimal Threshold under Perfect Foresight

In this section, we analyze the optimal threshold $\underline{r}$ in the absence of uncertainty over the natural rate in order to focus on the non-stochastic determinants of the central banker's optimal timing and a potential tendency to wait and see. So, let $\sigma = 0$ in the following.

Steady-State Utility First, we discuss the relationship between the optimal threshold given $\sigma = 0$ and the steady-state utility and show why it depends on the sign of u^{ss} in the following way (see also the illustrative scenarios in section 3.1):

$$\underline{r}\ |_{\sigma=0} \ = \begin{cases} g - \omega + \frac{\theta}{\delta} \cdot u^{ss}\ (=\ \hat{r}), & \text{if } u^{ss} < 0 \\ g - \omega, & \text{if } u^{ss} \geq 0. \end{cases} \tag{26}$$

In principle, optimal timing under perfect foresight does only depend on the anticipated path of period utility and thus on the anticipated path of the natural rate. With regard to the dependence of the optimal threshold given $\sigma = 0$ on the steady state of period utility, there are basically two scenarios of interest: one where the natural rate is initially below, and one where it is initially above its steady state:

Consider first the scenario where the natural rate is initially below its steady state, i.e., where $r_0 = r < r^{ss}$, and where period utility thus decreases in time. In this case, the calling-in move is either made immediately or never – there is nothing to wait for (as discussed in section 3.1). Clearly, it is optimal to never make the move if $r > \hat{r}$ which in a world where the natural rate is below its steady state can only occur if utility has a negative steady state.[41] However, it is optimal to make the move immediately if $r \leq \hat{r}$. If r_t is increasing and u_t thus decreasing in time waiting would involve the loss of potential benefits. In this case, the present value of the stream of period utilities given that the option is exercised in some future period t, $F(r_0 = r, T = t)$, is a decreasing function of time. Thus, if the natural rate is below its steady state, it is optimal to exercise the calling-in option immediately if $r \leq \hat{r}$ and thus if the exercise payoff $\mathcal{V}(r)$ is non-negative – regardless of whether the utility has a positive or a negative steady state.

[41] Recall that if utility has a positive steady state $r > \hat{r} = (1 + \frac{\theta}{\delta})u^{ss} + r^{ss}$ would require that $r > r^{ss}$.

A calling-in move at a natural rate above its steady state will generally only be made if $u^{ss} > 0$. In this case, the break-even threshold $\hat{r}$ is relatively high which reflects that the central banker would in principle accept short-run losses, i.e., negative period utilities, during some first phase and make the calling-in move at any $r_0 \leq \hat{r}$ if she lacked the flexibility to time the move freely.[42] If the flexibility to choose the optimal timing exists, those potentially acceptable losses can be avoided by deferring the calling-in move until the natural rate is sufficiently low such that positive period utilities will be realized. This is the case once the natural rate has fallen below the threshold $g - \omega$. So, if utility has a positive steady state, $\underline{r}|_{\sigma=0} = g - \omega$.

Speed of Mean-Reversion Equation (26) shows that the optimal threshold given $\sigma = 0$ only depends on the speed of mean-reversion θ if utility has a negative steady state such that $\underline{r}|_{\sigma=0} = \hat{r}$. However, as argued above, the threshold $\hat{r}$ is lower the faster the natural rate reverts to its steady state and thus the shorter the first phase is of the positive period utilities the central banker receives after the calling-in move. So, in order to accept such a shorter initial phase the central banker requires higher period utilities during that phase and thus a lower natural rate at the exercise of the calling-in option. Hence, as the origins of the blue lines in figure 1(a) for $u^{ss} < 0$ show, the optimal threshold is decreasing in θ.

The origins of the blue lines in figure 1(b) shows that the optimal threshold is independent of θ if the utility has a positive steady state. If $u^{ss} > 0$, a central banker with the ability to choose the timing of the calling-in move freely will wait until the natural rate hits the level $g - \omega$ below which natural rates imply positive period utilities.[43] Obviously, $g - \omega$ is independent of θ.

4.4 Determinants of the Optimal Threshold under Uncertainty

In the following, we remove the previous section's restriction and assume that $\sigma > 0$ to discuss the effects of uncertainty over the natural rate on the optimal threshold $\underline{r}$. Essentially, uncertainty over the future path of the natural rate and thus about the future utility of making the calling-in move adds value to the flexibility to time the move freely

[42] Recall that if $r > r^{ss}$ period utility will increase as time evolves.

[43] Recall that $g - \omega > r^{ss}$ if $u^{ss} > 0$. So, expected period utility is monotonically increasing in time for natural rates above their steady state.

and thus, in general, increases the central banker's tendency to wait and see. Intuition suggests that a higher tendency to wait and see from uncertainty due to a volatile natural rate will be reflected in a larger difference between the exercise threshold that would be optimal under perfect foresight and the generally optimal threshold. That is, intuition suggests that the length of the interval $[\underline{r}, \underline{r}|_{\sigma=0}]$ will be increasing in σ. We confirm this intuition for specific sets of parameter values numerically in section 5. In the following, we refer to the effect of uncertainty on the length of the interval above as *'variance effect'* (the variance, in turn, is increasing in σ and decreasing in θ).

However, the relationship between the *absolute level* of the optimal threshold $\underline{r}$ and uncertainty measured in terms of $\text{Var}(r_t)$ (for some fixed t), can be non-monotonic if utility has a negative steady state. The blue lines in figure 1(a) show this property and it is the speed of mean-reversion θ that accounts for this non-monotonicity. The reason is that θ affects two variables: the optimal threshold under perfect foresight ($\underline{r}|_{\sigma=0,u^{ss}<0} = \hat{r}$) and the tendency to wait and see. So, in addition to a variance effect, θ has a level effect through its impact on $\hat{r}$ which in turn determines the optimal threshold under perfect foresight if utility has a negative steady state. Since these two effects are of opposite signs, the overall effect of θ on $\underline{r}$ depends on the relative size of the level effect compared with the variance effect. With a parameter specification as in figure 1(a), the level effect will dominate for small σ whereas the variance effect will dominate if σ is large. If utility has a positive steady state, there will be no level effect, as shown by the blue lines in figure 1(b) where the optimal threshold is always increasing in θ for all $\sigma > 0$.

As opposed to this ambiguous relationship between the speed of mean-reversion and the optimal threshold, we show numerically for specific parameter constellations in the next section that a higher volatility parameter σ will reduce the level of $\underline{r}$ in these scenarios, regardless of whether θ is small or large.[44] The reason is that the exercise payoff the central banker requires in order to be willing to make the calling-in move under uncertainty is higher, the higher the extent of uncertainty is.[45] While the extent of uncertainty, in turn, is increasing in σ, the exercise payoff is larger, the lower the natural rate is at which large

[44] Alvarez and Dixit (2014) describe a non-monotonic relationship between a volatility parameter and a respective optimal threshold in a currency union's optimal stopping problem of breaking-up the union.

[45] Obviously, the requirement of a higher exercise payoff with increasing uncertainty is a main characteristic feature of equivalent financial/real option exercise problems (see, for instance, Dixit and Pindyck (1994, p. 153) in this regard).

notes will be called in (and thus the more relevant ELB-issues are compared to forgone seignorage revenues at a low natural-rate level).

4.5 Determinants of the Time Value of the Calling-In Option

In the following, we shed light on the time value of the calling-in option in different scenarios and its dependence on the parameters of the stochastic process that governs the natural rate: the volatility parameter σ and the speed of mean reversion θ. From a policy point of view, the question is simply: When is the time value large and thus the central banker's tendency to wait and see strong? Our statements in the following hold for the numerical examples we provide in section 5 but intuition suggests that they can be generalized to arbitrary parameter constellations. To obtain comparable results, we consider the value of the calling-in option measured at $r = \hat{r}$ in the different scenarios. At this point, the option has no intrinsic but just time value ($\mathcal{V}(\hat{r}) = 0$ implies that $V(\hat{r}) = TV(\hat{r})$).

Short-Term Natural Rate Volatility In the next section, we present some numerical illustrations for different scenarios to show that a higher volatility parameter σ adds time value to the calling-in option (see the blue lines in figure 2). The reason for this positive relationship between σ and TV is that a higher σ increases the probability that the natural rate will reach 'exceptionally' low levels in the future. At such 'exceptionally' low natural rate levels, ELB-issues will be highly relevant such that the central banker's benefits from calling in large banknotes will be 'exceptionally' large (for instance, because a very low natural rate level can imply that unconstrained optimal monetary policy rates lie far below the ELB). Exceptionally large benefits, in turn, are reflected by a relatively high exercise payoff $\mathcal{V}(\underline{r})$ and thus by a relatively high time value the calling-in option has at natural rates that lie above the optimal threshold $\underline{r}$.

Speed of Mean Reversion The speed of mean reversion θ affects the calling-in option's time value through two channels: through a mean-reversion channel and through an uncertainty channel. The uncertainty channel describes the impact θ has on the TV through its effect on the variance of the natural rate in future periods and thus on uncertainty over the future path of the natural rate. For $\sigma > 0$, the variance of r_t is a decreasing function of θ (as already noted in section 4.1). Analogously to the effect of σ on the TV, a lower speed of mean reversion θ and thus higher uncertainty over the future path of the natural rate adds time value to the calling-in option. However, the speed of mean-reversion also has a negative effect on the time value through the mean-reversion channel such that the overall effect of θ on the TV depends on whether it is the uncertainty or the mean-reversion effect that dominates.

In what follows, we want to isolate the mean-reversion channel and point out how exactly θ affects the TV at $r = \hat{r}$ through this channel. For this purpose we assume, for the moment, perfect foresight and let $\sigma = 0$: Whether the calling-in option has time value due to mean-reversion or not in a $\sigma = 0$-setting depends on whether the natural rate reverts to its steady state from above or from below. As argued above, if $r_0 = r < r^{ss}$, there will be nothing to wait for the central banker. In this case, the path of period utility is decreasing in time and thus optimality requires the central banker to make the calling-in move either immediately (if $r \leq \hat{r}$) or never (otherwise). If we consider $r = \hat{r}$, the existence of time value at this point will just depend on whether $\hat{r} \lesseqgtr r^{ss}$. With $\hat{r} = g - \omega + \frac{\theta}{\delta}(g - r^{ss} - \omega) = g - \omega + \frac{\theta}{\delta}u^{ss} = (1 + \frac{\theta}{\delta})u^{ss} + r^{ss}$ showing that $\hat{r}$ is greater (less) than r^{ss} if utility has a positive (negative) steady state, it is clear that the calling-in option will have time value at $r = \hat{r}$ due to mean reversion only if $u^{ss} > 0$ (where $\underline{r}|_{u^{ss}>0} < \hat{r}|_{u^{ss}>0}$). Moreover, and obviously, if there is time value at $\hat{r}$ it will be increasing in the steady state utility.

Since the mean-reversion channel will be relevant only if utility has a positive steady state, as argued above, it is sufficient to consider a perfect-foresight scenario where $u^{ss} > 0$: In such a scenario, the calling-in option's time value at $\hat{r}$ is increasing in θ. Two effects of a large θ contribute to a larger time value: Firstly, a higher speed of mean reversion shortens the time it takes for the natural rate to decline from $\hat{r}$ to $\underline{r}$ (although $\hat{r}$ increases in θ).[46] And secondly, a large θ increases the payoff from exercising the calling-in option

[46] To see this, consider $T^* = \frac{1}{\theta}\ln\left(\frac{r_0 - r^{ss}}{g - r^{ss} - \omega}\right)$ which for $r_0 = \hat{r} = (1 + \frac{\theta}{\delta})u^{ss} + r^{ss}$ becomes $T^* = \frac{1}{\theta} \cdot \ln\left(\frac{\delta + \theta}{\delta}\right)$. It is easy to show that $\frac{\partial T^*}{\partial \theta} = \frac{1}{\theta(\delta + \theta)} - \frac{\ln\left(\frac{\delta+\theta}{\delta}\right)}{\theta^2} < 0$ by making the substitution $x = \frac{\delta + \theta}{\delta}$ and then using the mean value theorem to show that $1 - \frac{1}{x} < \ln(x)$ for all $x > 1$.

at $\underline{r}$.[47] With $V(\hat{r}) = F(\hat{r}, T^*) = \mathcal{V}(\underline{r}) \cdot \exp(-\delta T^*) = TV(\hat{r})$ it is clear that a larger θ (in a $\sigma = 0$-scenario) increases the calling-in option's time value at $\hat{r}$ through its effects on the optimal exercise time and the exercise payoff.

To conclude, if utility has a positive steady state, the sign of the overall effect of θ on the TV, in general, will depend on the extent of the short-term volatility as captured by σ. In the next section, we show numerically for specific parameter values that while the effect of θ through the mean-reversion channel will dominate for small values of σ, the effect of θ through the uncertainty channel will dominate for large values of σ. The blue lines in figure 2(b) illustrate for a specific set of parameter values in a world where utility has a positive steady state, i.e., where $u^{ss} > 0$, that the time value of the calling-in option is increasing in θ for small σ while it is decreasing in θ for large σ. In contrast, if utility has a negative steady state which implies that only the uncertainty channel will be effective, the overall effect of an increase in θ will be to decrease the calling-in option's time value, as illustrated by the blue lines in figure 2(a).

5 Numerical Illustrations

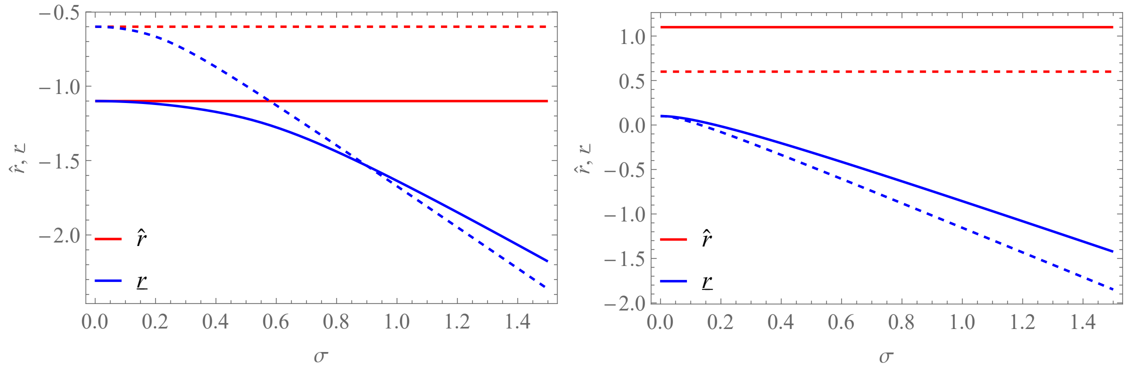
In what follows, we present and discuss several numerical examples for the central banker's wait-and-see tendency in different scenarios, i.e., for specific parameter values. Our claim is that the length of the wait-and-see region $[\underline{r}, \hat{r}]$ together with the time value of the calling-in option at some $r \in [\underline{r}, \hat{r}]$ in the single scenarios can be used to assess the relative likelihoods that wait-and-see behavior might actually be observed in corresponding scenarios in practice. The results in figures 1 to 3, and tables 1 and 2 have been obtained with the computer algebra system *Mathematica* by solving equation (25) for $\underline{r}$ and using (24) to evaluate $V(r)$ as defined in (23). Figure 1 shows how the break-even threshold $\hat{r}$ and the optimal threshold $\underline{r}$ depend on the steady-state utility u^{ss}, on the speed of mean-reversion θ, and on the volatility parameter σ. Figure 2 shows how the value V of

[47] To see this, note that at the point in time when the calling-in option is optimally exercised (which for $\sigma = 0$ is reached when $\underline{r} = g - \omega$), the natural rate rate will be still above its steady state. So, period utility from that point on is reverting to its steady state from below. The faster this reversion, the larger the present value of the stream of future period utilities. Hence, the exercise payoff at $\underline{r}$ is increasing in θ. Formally, this is obvious with $\mathcal{V}(\underline{r}) = F(r = g - \omega, T = 0) = \frac{1}{\delta}(g - r^{ss} - \omega) - \frac{1}{\delta+\theta}(g - r^{ss} - \omega) = \frac{\theta}{\delta(\delta+\theta)} u^{ss}$ and $\frac{\partial \mathcal{V}(\underline{r})}{\partial \theta} = \frac{1}{(\delta+\theta)^2} u^{ss} > 0$.

the calling-in option measured at $r = \hat{r}$ (compared to the exercise payoff $\mathcal{V}$) depends on u^{ss}, θ, and σ. Figure 3 shows how the exercise payoff of the calling-in option evaluated at $r = \underline{r}$ where $\mathcal{V}(\underline{r}) = V(\underline{r})$ depends on these parameters. For the sake of clarity, tables 1 and 2 show some values of $V(\hat{r})$ and $\mathcal{V}(\underline{r})$ for selected parameter constellations.

Scenarios with Negative Steady-State Utility For the regions of the parameter space specified below we show numerically that the length of the interval $[\underline{r}, \hat{r}]$, the time value of the calling-in option at $\hat{r}$, as well as the exercise payoff at $\underline{r}$ will increase in σ and decrease in θ if utility has a negative steady state (see also the argumentation in section 4). So there will be little room for wait-and-see behavior if uncertainty over the future path of the natural rate is relatively low. This implies that the natural rate of interest does not have to be at levels that are far below the break-even threshold to induce the central banker to make the calling-in move. In the numerical examples given in tables 1 and 2 such scenarios (i.e., scenarios with relatively low uncertainty over the future path of the natural rate in a world where calling in large notes involves long-run losses for the central banker) are described for $u^{ss} < 0$, $\sigma \in \{0.01,\ 0.5\}$, and $\theta \in \{0.5,\ 1\}$. While table 1 shows that the time value of the calling-in option measured at the break-even threshold will be relatively small in the aforementioned scenarios, table 2 shows that the net benefits the central banker will require to make the calling-in move in these scenarios are close to or just slightly above zero. Thus, in corresponding scenarios in practice, the move could be made as soon as the natural rate is so low that eliminating large banknotes will have net benefits that are just slightly above zero. A severe ELB-episode during or in the aftermath of a large-scale financial or economic crisis could be such a situation. Calling in large banknotes would be rational in such a scenario if the short-run benefits exceeded the long-run losses and waiting to make the move implied losing short-run benefits.

Scenarios with Positive Steady-State Utility In section 4, we discuss the non-monotonic relationship between uncertainty over the future path of the natural rate of interest (as reflected in $\mathrm{Var}(r_t)$) and the time value of the calling-in option in a world where the utility from calling in large notes has a positive steady state. We also describe the behavior of the break-even threshold and the optimal threshold which will both increase in



(a) negative steady-state utility

(b) positive steady-state utility

Figure 1: Break-even threshold $\hat{r}$ (red lines) and optimal threshold $\underline{r}$ (blue lines) for high (solid lines) and low (dotted lines) speed of mean reversion (parameter values (a): $r^{ss} = 0,\ g - \omega = -0.1,\ \delta = 0.1$; parameter values (b): $r^{ss} = 0,\ g - \omega = 0.1,\ \delta = 0.1$; solid lines: $\theta = 1$, dotted lines: $\theta = 0.5$).

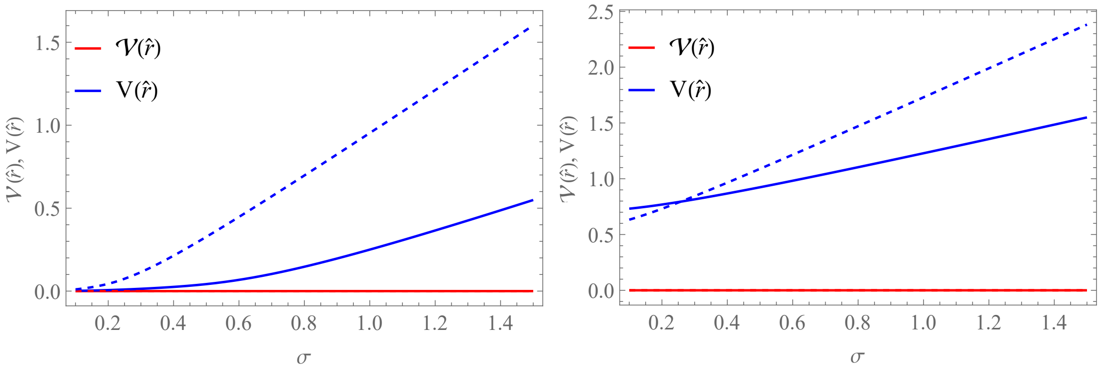
Time value of calling-in option at break-even threshold $V(\hat{r}) = TV(\hat{r})$						
	$\sigma = 0.01$		$\sigma = 0.5$		$\sigma = 1$	
u^{ss}	$\theta = 0.5$	$\theta = 1$	$\theta = 0.5$	$\theta = 1$	$\theta = 0.5$	$\theta = 1$
-0.2	≈ 0	≈ 0	0.148	0.019	0.654	0.084
-0.1	≈ 0	≈ 0	0.327	0.042	0.953	0.25
0	0.013	0.007	0.658	0.335	1.317	0.67
0.1	0.583	0.715	1.089	0.924	1.73	1.228
0.2	1.165	1.431	1.573	1.582	2.178	1.847

Table 1: Numerical solutions of the calling-in option's value at $r = \hat{r}$ for different values of the steady-state utility u^{ss}, the volatility parameter σ, and the speed of mean-reversion θ with $r^{ss} = 0$ and $\delta = 0.1$. All values are rounded to three decimal places.

Exercise payoff at optimal threshold $\mathcal{V}(\underline{r})$							
		$\sigma = 0.01$		$\sigma = 0.5$		$\sigma = 1$	
u^{ss}	$\mathcal{U}$	$\theta = 0.5$	$\theta = 1$	$\theta = 0.5$	$\theta = 1$	$\theta = 0.5$	$\theta = 1$
-0.2	-2	≈ 0	≈ 0	0.356	0.052	1.324	0.212
-0.1	-1	≈ 0	≈ 0	0.662	0.106	1.787	0.487
0	0	0.023	0.011	1.164	0.53	2.328	1.061
0.1	1	0.835	0.91	1.784	1.28	2.926	1.778
0.2	2	1.668	1.818	2.469	2.107	3.567	2.56

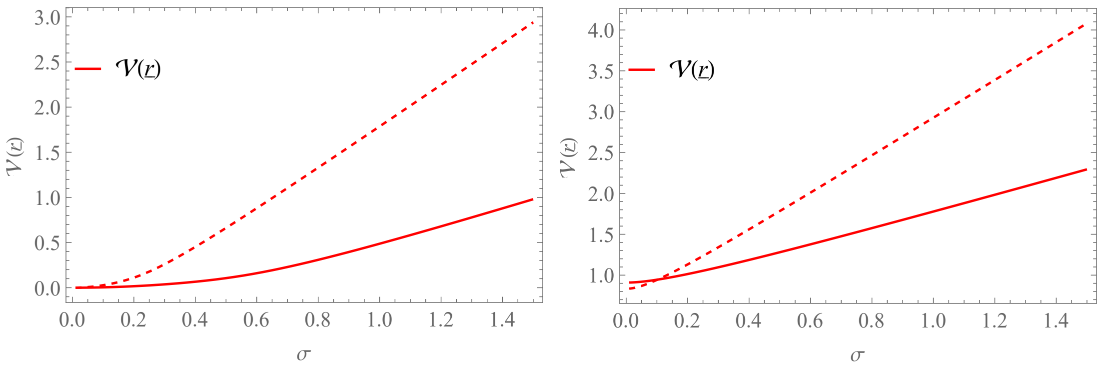
Table 2: Numerical solutions of the exercise payoff at $r = \underline{r}$ for different values of the steady-state utility u^{ss} (with $\mathcal{U}$ being defined as $\mathcal{U} = \int_0^\infty u^{ss} \cdot \exp(-\delta t)dt = \frac{1}{\delta}u^{ss}$), the volatility parameter σ, and the speed of mean-reversion θ with $r^{ss} = 0$ and $\delta = 0.1$. All values are rounded to three decimal places.

the speed of mean-reversion θ if $u^{ss} > 0$ – this means that the overall effect of a change in θ on the length of the wait-and-see region $[\underline{r},\ \hat{r}]$ depends on which bound of the interval reacts



(a) negative steady-state utility (b) positive steady-state utility

Figure 2: Exercise payoff $\mathcal{V}(\hat{r})$ (red lines) and option value $V(\hat{r})$ (blue lines) for high (solid lines) and low (dotted lines) speed of mean reversion (parameter values (a): $r^{ss} = 0,\ g - \omega = -0.1,\ \delta = 0.1$; parameter values (b): $r^{ss} = 0,\ g - \omega = 0.1,\ \delta = 0.1$; solid lines: $\theta = 1$, dotted lines: $\theta = 0.5$).



(a) negative steady-state utility (b) positive steady-state utility

Figure 3: Exercise payoff $\mathcal{V}(\underline{r})$ for high (solid lines) and low (dotted lines) speed of mean reversion (parameter values (a): $r^{ss} = 0,\ g - \omega = -0.1,\ \delta = 0.1$; parameter values (b): $r^{ss} = 0,\ g - \omega = 0.1,\ \delta = 0.1$; solid lines: $\theta = 1$, dotted lines: $\theta = 0.5$).

more sensitively to a change in θ. Taken all together, these properties imply that under special circumstances there will be more room for wait-and-see behavior if uncertainty over the natural rate is low (and not high, as intuition suggests). The numerical examples given in figure 1(b) and tables 1 and 2 for $u^{ss} > 0$ and $\sigma = 0.01$ show this case: the length of $[\underline{r},\ \hat{r}]$, the time value at $\hat{r}$ and the exercise payoff at $\underline{r}$ are all greater the lower the uncertainty over the natural rate is (recall that $\mathrm{Var}(r_t)|_{\sigma=0.01,\ \theta=0.5} > \mathrm{Var}(r_t)|_{\sigma=0.01,\ \theta=1}$).

But beyond that, the numerical examples primarily illustrate that the room for wait-and-see behavior will in general be relatively large if utility has a positive steady state. Table 2 shows for $u^{ss} > 0$ that the net benefit the central banker will require to make the calling-in move under relatively high levels of uncertainty ($\sigma \geq 0.5$) is significantly greater than zero and even a multiple of the present value of the infinite stream of steady-state utility, $\mathcal{U}$, defined as $\mathcal{U} = \int_0^\infty u^{ss} \cdot \exp(-\delta t) dt$. For $u^{ss} = 0.1$, $\sigma = 1$, and $\theta = 0.5$ the exercise payoff at the optimal threshold $\underline{r}$ is almost three times larger than the present value $\mathcal{U}$ of the infinite stream of steady-state utility. In practice, central banks could keep issuing large banknotes for a long time in a corresponding scenario even though it may already be beneficial to eliminate them immediately.

6 Conclusion

The goal of this paper is to stimulate research on optimal timing issues associated with any plans to phase out large-denomination banknotes. This research is essential because the debate on such plans is incomplete if it is only concerned with the costs and benefits while any timing aspects and a potential wait-and-see component of an optimal timing strategy in an uncertain economic environment are ignored. We condense the stochastic state of the economy into the natural rate and employ an optimal stopping model as a framework to explore such timing issues and to rationalize a central banker's wait-and-see tendency. The purpose of this approach is not to determine the exact empirical magnitudes of our results but to make clear that the stochastic properties and the expected path of the natural rate can be used as a first rough indicator of wait-and-see behavior in practice. In concrete terms, this means that the volatility and the expected path of the natural rate can be used to gauge whether the issuance of large notes could (in a positive dimension) or should (in a normative dimension) continue for years or decades even if the expected net benefits from calling them in right now were already greater than zero.

We use several numerical examples to illustrate states of the world where an optimal timing strategy to call in large notes involves or, on the other hand, does not involve a wait-and-see component. The central banker's tendency to wait and see is largest in scenarios with three key features. In particular, these are scenarios where 1) the long-run net benefits from calling in large banknotes are greater than zero which is the case if, in the long run, ELB-issues outweigh lost seignorage revenues; 2) the expected path of the natural rate is decreasing, implying that ELB-issues will become more and more relevant as time evolves; 3) the natural rate is highly volatile such that there is much uncertainty over the occurrence and duration of ELB-episodes in the near future. For corresponding scenarios, our findings indicate that the net benefits a central banker requires to be willing to call in large notes will by far be greater than zero. This can imply that the option to call in large banknotes will not be exercised until the natural rate has fallen to an exceptionally low level.

On the other hand, an optimal timing strategy involves no or only a small wait-and-see component in scenarios where 1) the long-run benefits from calling in large notes are negative; 2) the natural rate is expected to follow an increasing path; 3) its volatility is relatively low, so that the near future is less uncertain. Such states of the world are scenarios where ELB-issues become less and less relevant relative to considerations on seignorage losses as time evolves. Large banknotes will thus only be called in in times of extremely low natural rates implying a severe ELB-episode. Such situations could emerge in the course of pronounced recessions or in the aftermath of large-scale economic crises where monetary policy remains stuck at the ELB.

On the basis of our simple model, a guesstimate of the ECB's stance on plans to stop the issuance of the 200- or the 100-euro note is not difficult to divine: It is natural to assume that the ECB – if ever – will only make such a move during a severe ELB-episode. Of course, this speculation is highly sensitive to the assumptions we make. Any serious forecast in this regard will require at least a large-scale macro model as a framework to analyze timing issues associated with calling-in plans. Our model is highly stylized and has left out a number of factors that could change our results. For instance, the specification with the mean-reverting Ornstein-Uhlenbeck process we chose can easily be extended by assuming that the natural rate follows a jump diffusion. This could account for the hypothesis that large-scale economic or financial crises can lead to exceptional drops in the natural rate in their immediate aftermath (as indicated by the findings of Holston, Laubach, and Williams (2017) for the global financial crisis). We have also ignored that there is far more than one dimension of uncertainty. It is not only the future path of the natural rate that is unknown, but also the natural rate itself and its historic path. The reason is that the natural rate must be estimated and cannot be measured directly.[48]

[48] See, for instance, Weber, Lemke, and Worms (2008, section 5).

Moreover, there is also much uncertainty regarding the costs and benefits from calling in large notes. It could be hard to quantify them precisely. Another crucial assumption we make is that the central banker has full flexibility to time a calling-in move freely. A central banker approaching the end of her term in office (or a central bank's decision-making body shortly before its members change) could be driven by a precautionary motive from the fear that her successor will have a different objective function (another issue is a potential intervention of the government which could be driven by its own objective function and could try to change the denominational structure of banknotes by law). A number of further research questions will arise from allowing for the possibility of making sequential calling-in moves (e.g., to stop the issuance of the 100-euro note at a later date than the issuance of the 200-euro note) as well as from considering strategic interactions between central bankers in a multi-country setting. We leave the analysis along these dimensions for further research.

Appendix

A Detailed Solution of the Central Banker's Optimal Stopping Problem

A.1 Derivation of the Bellman Equation written as Ordinary Differential Equation

We show how to derive the Bellman ordinary differential equation (18) starting from equation (17). In technical regards, we refer to Dixit and Pindyck (1994, pp. 140–141) where the single steps we have to take are pointed out (note, that the differential equation in Dixit and Pindyck (1994, p. 140, equation (8)) results from geometric Brownian motion and thus, obviously, differs from (18)).

For the derivation of (18) that follows we use that

$$(dr_t)^2 = (dr_t) \cdot (dr_t) \tag{27}$$

$$= \theta^2 (r^{ss} - r_t)^2 (dt)^2 + 2 \cdot \theta (r^{ss} - r_t) dt \cdot \sigma dB_t + \sigma^2 (dB_t)^2 \tag{28}$$

$$= \sigma^2 dt, \tag{29}$$

which is obtained by using the rules $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$ and $dB_t \cdot dB_t = dt$ (recall that dr_t describes the dynamics of the mean-reverting Ornstein-Uhlenbeck process that governs the natural rate of interest with $dr_t = \theta(r^{ss} - r_t)dt + \sigma dB_t$ where B_t is Brownian motion and $\theta \in \mathbb{R}_{>0}$, $r^{ss} \in \mathbb{R}$, and $\sigma \in \mathbb{R}_{>0}$ are known constants).[49]

Now, let $V : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function for the value of the calling-in option dependent on the natural rate of interest r. Applying the Itô formula, using (3) and (29), we obtain

$$dV = V' dr_t + \frac{1}{2} V''(dr_t)^2 \tag{30}$$

$$= V' \cdot (\theta(r^{ss} - r_t)dt + \sigma dB_t) + \frac{1}{2} V'' \cdot \sigma^2 dt \tag{31}$$

with

$$\mathbb{E}[dV | r_t = r] = \theta(r^{ss} - r) \cdot V' dt + \frac{1}{2}\sigma^2 \cdot V'' dt \tag{32}$$

which is implied by $\mathbb{E}[dB_t] = 0$.[50] The Bellman ordinary differential equation (18) can now be obtained by using (32) to replace the right-hand side of (17).

A.2 Proof of Proposition 1

In section A.3 of this appendix, we point out *how* the solution of (18) can be obtained. In this section, we just prove proposition 1 and show first, that (23) solves the Bellman ODE (18), second, that this solution satisfies the right boundary condition and the non-negativity constraint, third, how to obtain c_1 in (24), and fourth, how to obtain the implicit definition of $\underline{r}$ in (25).

The first part of the proof is thus to show that

$$\frac{1}{2}\sigma^2 V'' + \theta(r^{ss} - r)V' - \delta V = 0 \tag{33}$$

[49]For the rules used to compute $(dr_t)^2$ see, for instance, Øksendal (2013, p. 45, equation (4.1.8)).

[50]Note that the Itô formula with the notation used here is given in Dixit and Pindyck (1994, pp. 79–81, 140). A discussion of the Itô formula in greater depth and in a general context is given, for instance, in Øksendal (2013, p. 44).

for

$$V(r) = c_1 \cdot H_{-\frac{\delta}{\theta}} \left(\frac{\sqrt{\theta}}{\sigma} \cdot (r - r^{ss}) \right) \tag{34}$$

where $H_\nu(z)$ denotes a Hermite function (see, for instance, Lebedev, 1965, p. 285).

We use the following representations of the first and second derivative of the Hermite function as given in Lebedev (1965, p. 289) with

$$H'_\nu(z) := \frac{\partial H_\nu(z)}{\partial z} = 2\nu H_{\nu-1}(z), \tag{35}$$

$$H''_\nu(z) := \frac{\partial^2 H_\nu(z)}{\partial z^2} = 2\nu H'_{\nu-1}(z), \tag{36}$$

to obtain

$$V'(r) := \frac{\partial V(r)}{\partial r} = c_1 \cdot 2 \cdot \left(-\frac{\delta}{\theta} \right) \cdot H_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma} \cdot (r - r^{ss}) \right) \cdot \frac{\sqrt{\theta}}{\sigma} \tag{37}$$

$$= -2c_1 \frac{\delta}{\sigma\sqrt{\theta}} H_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma} \cdot (r - r^{ss}) \right) \tag{38}$$

and

$$V''(r) := \frac{\partial^2 V(r)}{\partial r^2} = -2c_1 \frac{\delta}{\sigma\sqrt{\theta}} \cdot H'_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma} \cdot (r - r^{ss}) \right) \cdot \frac{\sqrt{\theta}}{\sigma} \tag{39}$$

$$= -2c_1 \frac{\delta}{\sigma^2} H'_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma} \cdot (r - r^{ss}) \right). \tag{40}$$

Now, we can use (34), (38), and (40) to reformulate the left-hand side of the Bellman ODE (33) and then show that this expression is in fact zero by making the following transformations:

$$\frac{1}{2}\sigma^2 \cdot (-2)c_1 \frac{\delta}{\sigma^2} H'_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma}(r - r^{ss}) \right) + \theta(r^{ss} - r) \cdot (-2)c_1 \frac{\delta}{\sigma\sqrt{\theta}} H_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma}(r - r^{ss}) \right)$$

$$- \delta c_1 H_{-\frac{\delta}{\theta}} \left(\frac{\sqrt{\theta}}{\sigma}(r - r^{ss}) \right) = 0 \quad \Big| \cdot \frac{1}{c_1} \tag{41}$$

$$\Leftrightarrow \; -\delta H'_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma}(r - r^{ss}) \right) - 2\theta(r^{ss} - r) \frac{\delta}{\sigma\sqrt{\theta}} H_{-\frac{\delta}{\theta}-1} \left(\frac{\sqrt{\theta}}{\sigma}(r - r^{ss}) \right)$$

$$
-\delta H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma}(r-r^{ss})\right)=0 \quad \Big| \cdot \frac{2}{\theta} \tag{42}
$$

$$
\Leftrightarrow -2\frac{\delta}{\theta}H'_{-\frac{\delta}{\theta}-1}\left(\frac{\sqrt{\theta}}{\sigma}(r-r^{ss})\right)=2\frac{\sqrt{\theta}}{\sigma}(r^{ss}-r)\cdot 2\frac{\delta}{\theta}H_{-\frac{\delta}{\theta}-1}\left(\frac{\sqrt{\theta}}{\sigma}(r-r^{ss})\right)
$$

$$
+2\frac{\delta}{\theta}H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma}(r-r^{ss})\right) \quad \Big| \text{substituting } \nu := -\frac{\delta}{\theta} \text{ and } z := \frac{\sqrt{\theta}}{\sigma}(r-r^{ss}) \tag{43}
$$

$$
\Leftrightarrow 2\nu H'_{\nu-1}(z) = 2zH'_{\nu}(z) - 2\nu H_{\nu}(z) \tag{44}
$$

where the last transformation is obtained by applying (35) to the first term on the right-hand side of (43). As shown in Lebedev (1965, p. 289, equation (10.4.5)), equation (44) is true – thus (41) is true, too. This proves that (23) is a solution of the Bellman ODE (18).

Let us now show that (23) satisfies the right boundary condition (21) and the non-negativity constraint (22). To see that the non-negativity constraint is satisfied, consider the integral representation of the Hermite function for Re $\nu < 0$ as given in Lebedev (1965, p. 290, equation (10.5.2)) with

$$
H_\nu(z) = \frac{1}{2\Gamma(-\nu)}\int_0^\infty \exp(-t^2-2tz)t^{-\nu-1}dt, \tag{45}
$$

where $\Gamma(\cdot)$ is the Gamma function as defined in Lebedev (1965, p. 1). With $\nu = -\frac{\delta}{\theta} \in \mathbb{R}_{<0}$ and therefore $\Gamma(-\nu) \in \mathbb{R}_{>0}$ it becomes immediately clear that $H_\nu(z) > 0$ for all $z \in \mathbb{R}$, and thus that (23) is non-negative for all $c_1 \in \mathbb{R}_{>0}$ and for all $\frac{\delta}{\theta} > 0$. That c_1 as defined in (24) is in fact greater than zero is easy to see with the same argument.

To see that the right boundary condition is satisfied, consider the asymptotic representation of $H_\nu(z)$ for large $|z|$ as given in Lebedev (1965, p. 292, equation (10.6.6)) for the special case of $z \in \mathbb{R}$ with

$$
H_\nu(z) = (2z)^\nu\left[\sum_{k=0}^{n}\frac{(-1)^k}{k!}(-\nu)_{2k}(2z)^{-2k} + O(|z|^{-2n-2})\right], \tag{46}
$$

where $(-\nu)_0 = 1$ and $(-\nu)_{2k} = \frac{\Gamma(-\nu+2k)}{\Gamma(-\nu)} = (-\nu)(-\nu+1)\cdots(-\nu+2k-1)$ (see Lebedev, 1965, p. 291). That (23) satisfies the right boundary condition for all $c_1 \in \mathbb{R}_{>0}$ and $\frac{\delta}{\theta} > 0$ follows directly from (46) for $\nu < 0$ (to see this, let $n = 0$ and consider $H_\nu(z) = (2z)^\nu\left[1 + O(|z|^{-2})\right]$ for $\nu < 0$).

To see how c_1 as defined in (24) is obtained, consider the smooth-pasting condition (20) which requires that $V'(\underline{r}) \overset{!}{=} \frac{\partial \mathcal{V}(\underline{r})}{\partial \underline{r}}$. Using (38) and

$$\frac{\partial \mathcal{V}(\underline{r})}{\partial \underline{r}} = -\frac{1}{\delta + \theta}, \tag{47}$$

(which is obtained by differentiating $\mathcal{V}$ as defined in (10)), the smooth-pasting condition can be rearranged to obtain c_1 as in (24).

To see how the implicit definition of $\underline{r}$ in (25) is obtained, consider the value-matching condition (19) requiring that $V(\underline{r}) \overset{!}{=} \mathcal{V}(\underline{r})$. The left-hand side of equation (25) is directly obtained by using (23) for V with c_1 in the representation given by (24). The right-hand side of equation (25) is just the exercise payoff as defined in (10) evaluated at $\underline{r}$. □

A.3 Solution of the Bellman Ordinary Differential Equation

Let us now outline how we obtained a solution of the central banker's optimal stopping problem and thus of the free boundary problem of solving the Bellman ODE (18) and finding $\underline{r}$. As a first step, we used the computer algebra system *Mathematica* (see Wolfram Research, Inc., 2015) to obtain a solution of (18) by using the *Mathematica*-function `DSolve`. *Mathematica* returned two linearly independent solutions – one of them being (23), the other one being a Kummer confluent hypergeometric function. After having proved that (23) solves (18) as outlined in section A.2 of this appendix, and since we do not need to prove uniqueness of our solution, we chose (23) to solve the central banker's optimal stopping problem.

To see that this solution is in fact correct, we can use the argument $\frac{\sqrt{\theta}}{\sigma} \cdot (r - r^{ss})$ of the Hermite function returned by Mathematica in order to transform (18) into a canonical form and then look up in Lebedev (1965) for the solution of this differential equation.[51] To transform (18), we use the technique outlined by Dixit and Pindyck (1994, p. 163) and accordingly introduce

$$z(r) = \frac{\sqrt{\theta}}{\sigma}(r - r^{ss}) \text{ with } z'(r) = \frac{\sqrt{\theta}}{\sigma}, \tag{48}$$

[51] See, for instance, Suzuki (2016, p. 35, equation (2)) for an equivalent substitution.

and use a function $w(z)$ to substitute

$$V(r) = w(z) \text{ with } V'(r) = w'(z(r)) \cdot z'(r) = \frac{\sqrt{\theta}}{\sigma} w'(z) \text{ and } V''(r) = \frac{\theta}{\sigma^2} w''(z). \tag{49}$$

Therewith, and with $r = \frac{\sigma}{\sqrt{\theta}} z + r^{ss}$, equation (18) can be transformed into

$$\frac{1}{2}\sigma^2 \frac{\theta}{\sigma^2} w''(z) + \theta \left(r^{ss} - (\frac{\sigma}{\sqrt{\theta}} z + r^{ss}) \right) \frac{\sqrt{\theta}}{\sigma} w'(z) - \delta w(z) = 0, \tag{50}$$

which can be simplified to

$$w'' - 2zw' + 2\nu w = 0, \tag{51}$$

where $\nu := -\frac{\delta}{\theta} < 0$. Now, this differential equation, its solutions, and the Hermite function are discussed in great detail in Lebedev (1965, pp. 283-299).

The general solution of (51) is stated by Lebedev (1965, p. 286, equation (10.2.17)) and reads

$$w = M H_\nu(z) + N \exp(z^2) H_{-\nu-1}(\mathbb{i}z), \tag{52}$$

where M, N are constants, $\mathbb{i}^2 = -1$, and $H_\nu(z)$ is a Hermite function (the definition of a Hermite function is given by Lebedev (1965, p. 285)).

Re-substituting with $w = V$, $z = \frac{\sqrt{\theta}}{\sigma}(r - r^{ss})$ and $\nu = -\frac{\delta}{\theta}$, the general solution of the Bellman equation (18) can be written as

$$\begin{aligned} V(r) = {} & c_1 H_{-\frac{\delta}{\theta}} \left(\frac{\sqrt{\theta}}{\sigma} (r - r^{ss}) \right) \\ & + c_2 \exp\left(\left(\frac{\sqrt{\theta}}{\sigma} (r - r^{ss}) \right)^2 \right) H_{\frac{\delta}{\theta} - 1} \left(\mathbb{i} \left(\frac{\sqrt{\theta}}{\sigma} (r - r^{ss}) \right) \right), \end{aligned} \tag{53}$$

where c_1, c_2 are constants. As argued above, we now let $c_2 = 0$.[52] This yields a particular solution that satisfies the right boundary condition (21) and the non-negativity constraint (22). As also shown above, (19) and (20) can now be used to determine c_1 and $\underline{r}$.

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[52]See also, for instance, Parlour and Walden (2009, pp. 14-15), Garlappi and Yan (2011, p. 819), or Suzuki (2016, p. 39) who have equivalent/similar differential equations and obtain equivalent/similar solutions.

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Heinrich-Heine-University of Düsseldorf

Department of Economics

Universitätsstraße 1 _ 40225 Düsseldorf
www.dice.hhu.de

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